**Group:** — Let $G$ be a non-empty set and $O$ is binary operation then the system $(G, O)$ is called group iff the following postulates are satisfied:

(i) **closure law:** — For all $a, b \in G$ such that
$a \, O \, b \in G$.

i.e. $G$ is closed under the binary operation $O$.

(ii) **Associative law:** — For all $a, b, c \in G$ such that
$(a \, O \, b) \, O \, c = a \, O \, (b \, O \, c)$

i.e. associative law for the operation $O$ holds.

(iii) **Existence of inverse elements:** — For every $a \in G$ there exists an element $b \in G$ such that $a \, O \, b = b \, O \, a = e$, $e$ being the identity element

i.e. the inverse elements of all the elements of $G$ exist. The inverse of $a$ denoted by $a^{-1}$.

(iv) **Existence of identity element:** — For all $a \in G$ and there exists an element $e \in G$ such that $a \, O \, e = e \, O \, a = a$.

i.e. the identity element exist in $G$.

**Theorem :—** Show that the identity element in a group is unique.

or

Show that a group has a unique neutral element.

**Proof:** — Let suppose that a group $(G, O)$ has two identity elements $e$ and $e'$, where $e \neq e'$.

When $e$ is identity element then from the law of existence of identity element of a group, we have

$$a \, O \, e = e \, O \, a = a \quad \underline{\qquad} \quad (1) \ \forall \, a \in G$$

And when $e'$ is identity of $G$ then

$$a \, O \, e' = e' \, O \, a = a \quad \underline{\qquad} \quad (2) \ \forall \, a \in G$$

Here the equality of (1) and (2) show that
$$e = e'$$

This lead our contradiction that $e \neq e'$. Hence our supposition is wrong. So the identity element of a group is unique.

**2.**

**Theorem:** — Show that every element of a group has a unique inverse.

        or

    Prove that the inverse element of an element in a group is unique.

**Proof:** — Let $(4, 0)$ be a group and $a$ be any element of $4$. Suppose $x$ and $y$ are two inverses of $a$ then we have

$$a \circ x = x \circ a = e \quad [\because x \text{ is inverse of } a] \text{——(1)}$$

$$a \circ y = y \circ a = e \quad [\because y \text{ is inverse of } a] \text{——(2)}$$

from above two relations, we have

$$a \circ x = a \circ y = e$$

$$\Rightarrow x \circ (a \circ x) = x \circ (a \circ y) \quad [\text{multiply left by } x]$$

$$\Rightarrow [x \circ a] \circ x = (x \circ a) \circ y \quad [\text{by associative law}]$$

$$\Rightarrow e \circ x = e \circ y \quad (\text{by (1)})$$

$$\Rightarrow x = y \quad [\because e \text{ is identity}]$$

This shows that any two inverse of $a$ must be identical. Hence every element of a group $4$ is Unique.

—— Proved